Exercise 17

Use power series to solve the initial-value problem

$$y'' + xy' + y = 0 \quad y(0) = 0 \quad y'(0) = 1$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of n in the summand, the second series can start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Make the substitution $n = k + 2$ in the first series and the substitution $n = k$ in the second and third series.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^{(k+2)-2} + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

Simplify the first sum.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} x^k + k a_k x^k + a_k x^k \right] = 0$$

Simplify the summand.

$$\sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2}+(k+1)a_k\right]x^k=0$$

Since x^k isn't zero, the quantity in square brackets must be zero.

$$(k+2)(k+1)a_{k+2}+(k+1)a_k=0$$

Solve for a_{k+2}.

$$a_{k+2}=-\frac{1}{k+2}a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$\begin{aligned}
k=0:&\quad a_2=-\frac{1}{0+2}a_0=-\frac{a_0}{2}\\
k=1:&\quad a_3=-\frac{1}{1+2}a_1=-\frac{a_1}{3}\\
k=2:&\quad a_4=-\frac{1}{2+2}a_2=-\frac{1}{4}\left(-\frac{a_0}{2}\right)=(-1)^2\frac{a_0}{2\cdot 4}\\
k=3:&\quad a_5=-\frac{1}{3+2}a_3=-\frac{1}{5}\left(-\frac{a_1}{3}\right)=(-1)^2\frac{a_1}{3\cdot 5}\\
k=4:&\quad a_6=-\frac{1}{4+2}a_4=-\frac{1}{6}\left[(-1)^2\frac{a_0}{2\cdot 4}\right]=(-1)^3\frac{a_0}{2\cdot 4\cdot 6}\\
k=5:&\quad a_7=-\frac{1}{5+2}a_5=-\frac{1}{7}\left[(-1)^2\frac{a_1}{3\cdot 5}\right]=(-1)^3\frac{a_1}{3\cdot 5\cdot 7}\\
&\vdots
\end{aligned}$$

The general formula for the even subscripts is

$$a_{2m}=(-1)^m\frac{a_0}{(2m)!!}=(-1)^m\frac{a_0}{2^m m!},$$

and the general formula for the odd subscripts is

$$a_{2m+1}=(-1)^m\frac{1}{(2m+1)!!}a_1=(-1)^m\frac{2^m m!}{(2m+1)!}a_1.$$

Therefore, the general solution is

$$\begin{aligned}
y(x)&=\sum_{m=0}^{\infty}a_m x^m\\
&=\sum_{m=0}^{\infty}a_{2m}x^{2m}+\sum_{m=0}^{\infty}a_{2m+1}x^{2m+1}\\
&=\sum_{m=0}^{\infty}(-1)^m\frac{a_0}{2^m m!}x^{2m}+\sum_{m=0}^{\infty}(-1)^m\frac{2^m m!}{(2m+1)!}a_1x^{2m+1},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants.

Differentiate it with respect to x.

$$y'(x) = \sum_{m=1}^{\infty} (-1)^m \frac{a_0}{2^m m!}(2mx^{2m-1}) + \sum_{m=0}^{\infty} (-1)^m \frac{2^m m!}{(2m+1)!} a_1[(2m+1)x^{2m}]$$

Apply the initial conditions to determine a_0 and a_1.

$$y(0) = (-1)^0 \frac{a_0}{2^0 0!} = 0$$

$$y'(0) = (-1)^0 \frac{2^0 0!}{[2(0)+1]!} a_1[2(0)+1] = 1$$

Solving this system yields

$$a_0 = 0 \quad \text{and} \quad a_1 = 1.$$

Therefore, the solution to the initial value problem is

$$y(x) = \sum_{m=0}^{\infty} (-1)^m \frac{2^m m!}{(2m+1)!} x^{2m+1}.$$